The code and the graph rules of Dale recurrent networks

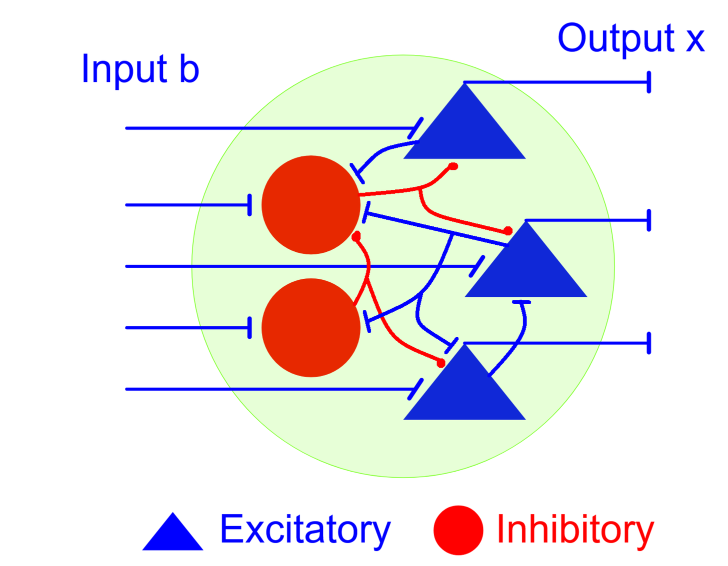
Nikola Milićević, Vladimir Itskov, Pennsylvania State University, Department of Mathematics,
nqm5625@psu.edu,vladimir.itskov@psu.edu

PennState Eberly College of Science

Overview

We describe the combinatorics of the fixed points and the steady states of a recurrent network that satisfies the Dale's law. With some natural condition on the spectrum of the synaptic matrix, this description requires only the connectivity features of the network.

Background



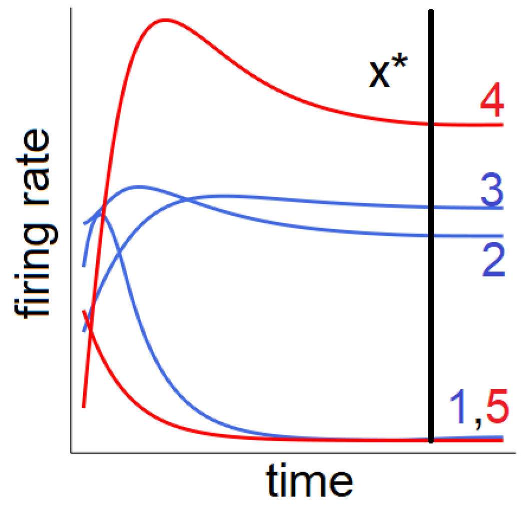
A rate model of a recurrent neural network, has the dynamics of the firing rates $x_i(t)$ described by

$$\tau\frac{dx_i}{dt} = -x_i + \left[\sum_{j=1}^{n} W_{ij}x_j + b_i\right]_+ ,$$

$[y]_+ = \max(0, y)$ is the ReLU function.

We assume the network respects Dale's law, where the neurons are either excitatory (denoted as $\mathcal{E}$) or inhibitory (denoted as $\mathcal{I}$). Following a common architecture of the neocortex, we assume that the excitatory neurons "broadcast" the output, while the activity of the inhibitory neurons is not observable *directly* from outside of the network.

The combinatorial code of a Dale network



Here $\mathrm{supp}_+ x^*$=23

For a firing rate vector $x = (x_1, \ldots, x_n)$, we consider the set of active excitatory neurons:

$$\mathrm{supp}_+ x = \{i \in \mathcal{E} \mid x_i > 0\} \subset \mathcal{E}.$$

A firing rate vector x^* is a fixed point of the network if $x(t) = x^*$ is a constant solution. A steady state x^* is an asymptotically stable fixed point.

The *combinatorial* code is the set of all possible patterns of excitatory neural activation at the fixed points.

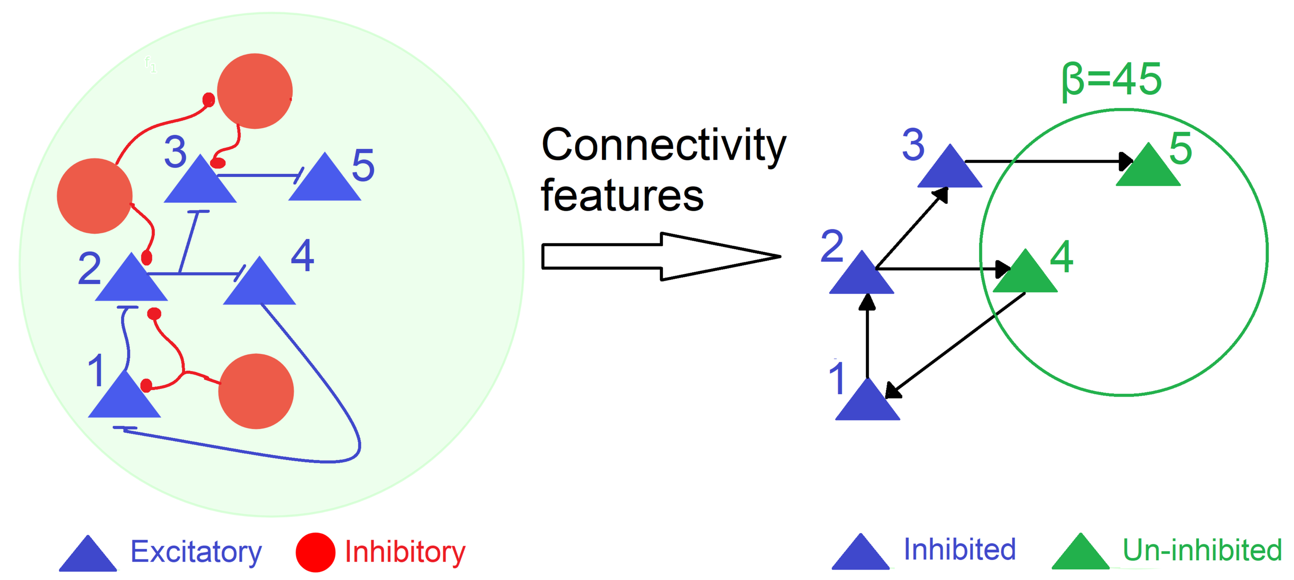
$$\mathcal{C}(W) \stackrel{\text{def}}{=} \bigcup_{b\in\mathbb{R}^n_{\geq 0}} \{\mathrm{supp}_+ x^* \mid x^* \in \mathbb{R}^n_{\geq 0} \text{ is a fixed point of the network}\}.$$

Similarly, the *stable combinatorial code* is the set of the excitatory supports of steady states:

$$\mathcal{SC}(W) \stackrel{\text{def}}{=} \bigcup_{b\in\mathbb{R}^n_{\geq 0}} \{\mathrm{supp}_+ x^* \mid x^* \in \mathbb{R}^n_{\geq 0} \text{ is a steady state of the network}\}.$$

The relevant connectivity features of Dale networks

Dale network — $G_{\mathcal{E}}$, the excitatory subnetwork



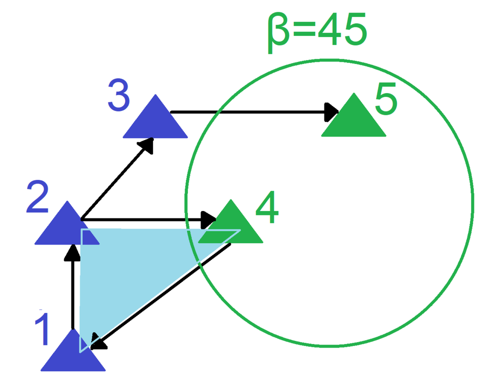
We find two connectivity features are of interest:

- The excitatory connectivity graph $G_{\mathcal{E}}$, whose vertices are the excitatory neurons $\mathcal{E}$ and whose directed edges correspond to synapses of the network among excitatory neurons.
- The un-inhibited set of excitatory neurons β.

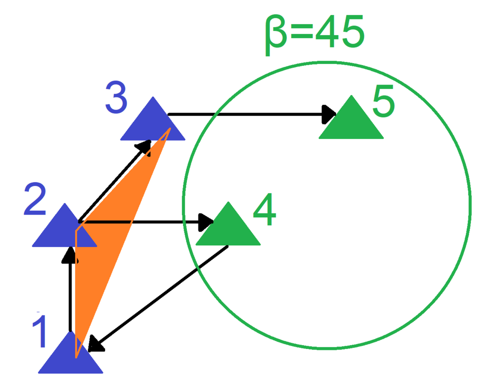
Only excitatory-excitatory and inhibitory-excitatory synapses matter for understanding the combinatorial code.
We combine these features into the following object:

$$\mathrm{code}(G_{\mathcal{E}}, \beta) \stackrel{\text{def}}{=} \{\sigma \subseteq \mathcal{E} \mid t(\sigma) \cap \beta \subset \sigma\},$$

where $t(\sigma)$ are the synaptic targets of σ in the excitatory subnetwork.



124 is in $\mathrm{code}(G_{\mathcal{E}}, \beta)$



123 is not in $\mathrm{code}(G_{\mathcal{E}}, \beta)$

The characterization of the combinatorial code

Connectivity and spectrum of synaptic weights completely determine the combinatorial code.

Theorem 1. A codeword σ is in the code $\mathcal{C}(W)$ if and only if the following both are satisfied:

1. (spectral condition) $\rho(W_{\beta_\sigma}) < 1$.
2. (connectivity feature) σ is in $\mathrm{code}(G_{\mathcal{E}}, \beta)$.

- β_σ are the neurons in σ that are un-inhibited, $\beta_\sigma = \sigma \cap \beta$.
- W_{β_σ} are the synaptic weights in the network among the neurons in β_σ.
- $\rho(W_{\beta_\sigma})$ is the spectral radius of W_{β_σ}.

The weak coupling regime

The weak coupling regime is the requirement that $||W||_F < 1$, where $||W||_F = \sqrt{\mathrm{trace}(W^TW)}$ is the Frobenius matrix norm. In the weak coupling regime all fixed points are steady states.

Theorem 2. In the weak coupling regime, the combinatorial code is completely determined by the connectivity features:

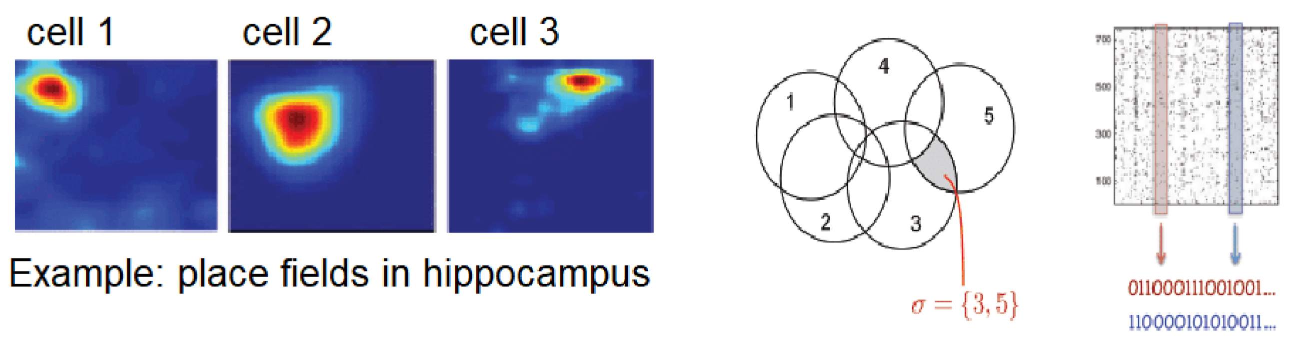
$$\mathcal{C}(W) = \mathcal{SC}(W) = \mathrm{code}(G_{\mathcal{E}}, \beta).$$

Furthemore, in the weak coupling regime the combinatorial code is a lattice. Given a code that is a lattice, we show how to construct a weak coupling network that outputs it.

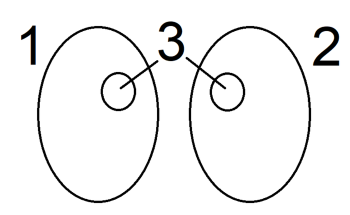
Theorem 3. For a code that is a lattice, there is a learning rule for constructing a Dale network that produces it.

Dale networks output convex codes

Neural activity in many sensory systems is organized by means of convex receptive fields. Neural codes that result from these receptive fields are constrained by convexity of the receptive fields, since not every neural code is compatible with convex receptive fields.



Example: place fields in hippocampus



The code {1,2,13,23} is not convex.

Question: How do neural circuits enforce the convexity of receptive fields?

Theorem 4. A generic recurrent neural network that satisfies the Dale's law outputs convex codes.

Reference

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